

THE LOEWY STRUCTURE OF THE PROJECTIVE INDECOMPOSABLE MODULES FOR A_{10} IN CHARACTERISTIC 3

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ABSTRACT. We compute the Loewy structure of the indecomposable projective modules for the group algebra FG , where G is the alternating group on 10 letters and F is an algebraically closed field of characteristic 3.

1. INTRODUCTION

Throughout, F is an algebraically closed field of characteristic 3. We denote each simple module for a group by its dimension, together with a subscript if there is more than one simple module of that dimension. Thus, the simple FA_{10} modules are denoted 1, 34, 41, 84, 224 (lying in the principal block B_0 of defect 4), 9, 36, 90, 126, 279 (in a block B_1 of defect 2), and 567 (a projective simple in the unique block B_2 of defect 0). Since blocks of defect 0 are easy to describe, we shall only be interested in the blocks of non-zero defect. We denote the central idempotents for the principal block and the block of defect 2 by e_0 and e_1 respectively. The central idempotent for the principal block of FA_9 is denoted f_0 .

Our methodology employs standard techniques such as Frobenius Reciprocity together with computation using the MeatAxe. The MeatAxe is used to find full or partial submodule lattices of certain induced modules, and we thus obtain sufficient information to determine the Loewy structures of the indecomposable projectives for FA_{10} . As a by-product of this process, we also obtain information about all the spaces $\text{Ext}_{A_{10}}^1(S, T)$. This can be viewed as the first step in constructing the Ext-quiver for this group. To our knowledge, no “reasonably nice” Ext-quivers of the alternating groups are in the published literature - one reason for this may be the difficulty of generating a consistent labelling for the simple modules.

The main results are now stated.

Theorem 1. *The Loewy structure of the projective indecomposable modules of B_0 are as follows:*

$$\begin{array}{ccccccc} & & & & 1 \\ & & & & 34 & 41 & 84 & 224 \\ & & & 1 & 1 & 1 & 1 & 34 & 41 & 41 & 84 & 224 & 224 \\ 1 & 1 & 1 & 34 & 34 & 34 & 41 & 41 & 41 & 84 & 84 & 224 & 224 \\ & & 1 & 1 & 1 & 1 & 34 & 41 & 41 & 84 & 224 & 224 \\ & & & & & & 34 & 41 & 84 & 224 \\ & & & & & & & & & 1 \end{array}$$

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$ \begin{array}{c} 34 \\ 1 \quad 41 \quad 224 \\ 1 \quad 34 \quad 34 \quad 41 \quad 84 \quad 224 \\ 1 \quad 1 \quad 1 \quad 34 \quad 41 \quad 41 \quad 84 \quad 224 \\ 1 \quad 34 \quad 34 \quad 41 \quad 84 \quad 224 \\ 1 \quad 41 \quad 224 \\ 34 \end{array} $							$ \begin{array}{c} 41 \\ 1 \quad 34 \quad 84 \quad 224 \\ 1 \quad 1 \quad 34 \quad 41 \quad 41 \quad 41 \quad 84 \quad 224 \\ 1 \quad 1 \quad 1 \quad 34 \quad 34 \quad 41 \quad 84 \quad 84 \quad 224 \quad 224 \\ 1 \quad 1 \quad 34 \quad 41 \quad 41 \quad 41 \quad 84 \quad 224 \\ 1 \quad 34 \quad 84 \quad 224 \\ 41 \end{array} $						
$ \begin{array}{c} 84 \\ 1 \quad 41 \quad 84 \\ 1 \quad 34 \quad 41 \quad 84 \quad 84 \quad 224 \\ 1 \quad 1 \quad 34 \quad 41 \quad 41 \quad 84 \quad 224 \\ 1 \quad 34 \quad 41 \quad 84 \quad 84 \quad 224 \\ 1 \quad 41 \quad 84 \\ 84 \end{array} $							$ \begin{array}{c} 224 \\ 1 \quad 34 \quad 41 \quad 224 \\ 1 \quad 1 \quad 34 \quad 41 \quad 84 \quad 224 \quad 224 \\ 1 \quad 1 \quad 34 \quad 41 \quad 41 \quad 84 \quad 224 \quad 22 \\ 1 \quad 1 \quad 34 \quad 41 \quad 84 \quad 224 \quad 224 \\ 1 \quad 34 \quad 41 \quad 224 \\ 224 \end{array} $						

Theorem 2. *The Loewy structure of the principal indecomposable modules of B_1 are as follows:*

$ \begin{array}{c} 9 \\ 126 \quad 279 \\ 9 \quad 9 \quad 36 \quad 90 \\ 126 \quad 279 \\ 9 \end{array} $				$ \begin{array}{c} 36 \\ 126 \quad 279 \\ 9 \quad 36 \quad 36 \quad 90 \\ 126 \quad 279 \\ 36 \end{array} $				$ \begin{array}{c} 90 \\ 126 \\ 9 \quad 36 \quad 90 \\ 126 \\ 90 \end{array} $			
$ \begin{array}{c} 126 \\ 9 \quad 36 \quad 90 \\ 126 \quad 126 \quad 279 \\ 9 \quad 36 \quad 90 \\ 126 \end{array} $				$ \begin{array}{c} 279 \\ 9 \quad 36 \\ 126 \quad 279 \\ 9 \quad 36 \\ 279 \end{array} $							

We note that the Loewy and socle series for the principal indecomposable modules (PIMs) over F are the same, which is also the case for A_6, A_7, A_8 and A_9 .

The Loewy structure of the PIMs for FA_6 is well-known and appears in (Benson, 1984); their module diagrams can be found in (Benson and Carlson, 1987). The structures for FA_7 and FA_8 have been calculated in (Scopes, 1988), while that for FA_9 has been done in (Siegel, 1991). The structures for FA_8 and FA_9 have also been done in characteristic two in (Benson, 1983a) and (Benson, 1983b) respectively. The results for FA_{10} in characteristic 3 have not been published to our knowledge.

If A is a group algebra over F and M a finitely generated A -module, write $L_i(M)$ and $S_i(M)$ for the i^{th} Loewy layer of M and the i^{th} socle layer of M , respectively. We shall write $(M, N)_A$ for $\dim_F \text{Hom}_A(M, N)$, M^* for $\text{Hom}_F(M, F)$ regarded as an A -module, $(M, N)_A^1$ for $\dim_F \text{Ext}_A^1(M, N)$, and P_M for the projective cover of M . We also denote a uniserial module of Loewy length 3 with head S , heart T and socle U by $\mathfrak{U}(S; T; U)$, and a module M with $L_1(M) = S_1, L_2(M) = S_2 \oplus S_3$ and $L_3(M) = S_4$ by $\mathfrak{D}(S_1; S_2, S_3; S_4)$ (here all the S_i are simple).

Facts about modular representation theory can be found in (Landrook, 1983). We use Brauer characters and GAP (GAP, 2013) to find the composition factors of the A_{10} -modules we study. We will occasionally use a (Benson-Carlson) module diagram to describe an A -module M . This is a finite directed graph with vertices labelled by simple modules, and with an edge from a vertex S to a vertex T corresponding to a non-trivial

$$\begin{array}{cccccc}
& & 41 & & & 35 \\
& & 1 & 7 & 35 & \\
1 & 7 & 21 & 41 & 41 & 35 \\
1 & 1 & 7 & 7 & 21 & 35 & 35 \\
& 1 & 7 & 21 & 41 & 41 & 35 \\
& & 1 & 7 & 35 & \\
& & 41 & & & \\
& & & & & 35 \\
& & & & & 1 & 7 & 21 & 41 & 35 \\
& & 1 & 1 & 7 & 7 & 21 & 41 & 35 & 35 & 35 \\
& & 1 & 1 & 7 & 7 & 21 & 41 & 35 & 35 & 35 \\
& & & & & 1 & 7 & 21 & 41 & 35 \\
& & & & & & & 35
\end{array}$$

ii) *Non-principal blocks:*

$$\mathfrak{U}(27; 189; 27) , \mathfrak{U}(189; 27; 189) , 162 .$$

3. INDUCTION AND RESTRICTION BETWEEN A_9 AND A_{10}

3.1. Restriction from A_{10} to A_9 .

Theorem 7.

$$\begin{aligned}
1_{A_{10}} \downarrow &= 1_{A_9} , \quad 9_{A_{10}} \downarrow = \mathfrak{U}(1; 7; 1) , \quad 34_{A_{10}} \downarrow = 7_{A_9} \oplus 27_{A_9} , \quad 36_{A_{10}} \downarrow = \mathfrak{D}(7; 1, 21; 7) , \\
41_{A_{10}} \downarrow &= 41_{A_9} , \quad 84_{A_{10}} \downarrow = \mathfrak{D}(21; 7, 35; 21) , \quad 90_{A_{10}} \downarrow = \mathfrak{D}(41; 1, 7; 41) , \\
126_{A_{10}} \downarrow &= \mathfrak{D}(35; 21, 35; 35) , \quad 224_{A_{10}} \downarrow = 35_{A_9} \oplus 189_{A_9} , \\
279_{A_{10}} \downarrow &= 13_{A_8} \uparrow^{A_9} \oplus 162_{A_9} = \mathfrak{U}(41; 35; 41) , \quad 567_{A_{10}} \downarrow = 162_{A_9} \oplus \mathfrak{U}(189; 27; 189) .
\end{aligned}$$

Proof. This follows easily from Frobenius Reciprocity, block theory and self-duality of all the modules involved. For example, using Brauer characters, $36_{A_{10}} \downarrow$ has composition factors $1_{A_9} + 2(7_{A_9}) + 21_{A_9}$. By Frobenius Reciprocity, $L_1(36 \downarrow) = S_1(36 \downarrow) = 7$. Hence, self-duality of $36 \downarrow$ gives the result. \square

3.2. Induction from A_9 to A_{10} .

Theorem 8.

$$\begin{aligned}
1_{A_9} \uparrow &= 1_{A_{10}} \oplus 9_{A_{10}} , \quad 7_{A_9} \uparrow = 34_{A_{10}} \oplus 36_{A_{10}} , \quad 21_{A_9} \uparrow = \mathfrak{D}(84; 1, 41; 84) , \\
35_{A_9} \uparrow &= 126_{A_{10}} \oplus 224_{A_{10}} , \quad 41_{A_9} \uparrow = 41_{A_{10}} \oplus 90_{A_{10}} \oplus 279_{A_{10}} , \\
162_{A_9} \uparrow &= P_{162_{A_9}} \uparrow = P_{162_{A_{10}}} \oplus P_{567_{A_{10}}} ,
\end{aligned}$$

$$\begin{array}{cccc}
& & 34 & & 224 \\
& & 1 & 41 & \\
27_{A_9} \uparrow = & 34 & 84 & , & 189_{A_9} \uparrow = 567_{A_{10}} \oplus & 1 & 34 & 84 & 224 \\
& & 1 & 41 & & & 1 & 41 & 224 \\
& & 34 & & & & & 224
\end{array}$$

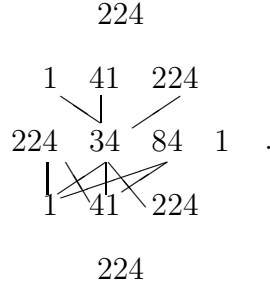
Proof. With the exception of $27 \uparrow$ and $189 \uparrow$, all structures follow from block theory, Frobenius Reciprocity, Brauer characters and self-duality.

For $27 \uparrow$ and $189 \uparrow$, the MeatAxe gives the claimed Loewy layers. Furthermore, the submodule lattice for $27 \uparrow$ implies that every simple module in $L_i(27 \uparrow)$ extends every simple module in $L_{i+1}(27 \uparrow)$.

For $189 \uparrow$, we only obtain partial information regarding extensions between modules in $L_3(189 \uparrow)$ and modules in both $L_2(189 \uparrow)$ and $L_4(189 \uparrow)$, namely,

- 34 extends 1s, 41s and 224s;
- 84 extends only 1s and 41s;
- 224 extends 1s and 41s (and may or may not extend 224s),
- 1 in $L_3(189 \uparrow)$ extends 41s in $L_2(189 \uparrow)$ and $L_4(189 \uparrow)$ (and possibly other simple modules).

A partial diagram for $189 \uparrow$ is as follows:



□

4. CALCULATION OF $(S, T)_{A_{10}}^1$ FOR SIMPLE FA_{10} -MODULES S AND T

4.1. Non-principal block.

Lemma 1.

$$(9, S)_{A_{10}}^1 = \begin{cases} 1, & \text{if } S \in \{126, 279\}; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We have

$$\begin{aligned}
 (9, S)_{A_{10}}^1 &= (1 \oplus 9, S)_{A_{10}}^1 && \text{by block theory,} \\
 &= (1 \uparrow_{A_9}^{A_{10}}, S)_{A_{10}}^1 && \text{by Theorem 8,} \\
 &= (1_{A_9}, S \downarrow_{A_9}^{A_{10}})_{A_9}^1 && \text{by Ext Reciprocity.}
 \end{aligned}$$

By Theorem 3, Theorem 5 and Theorem 7,

$$(1_{A_9}, 9 \downarrow_{A_9}^{A_{10}})_{A_9}^1 = (1_{A_9}, \mathfrak{U}(1; 7; 1))_{A_9}^1 = (1_{A_9}, 1 \uparrow_{A_8}^{A_9})_{A_9}^1 = (1_{A_8}, 1_{A_8})_{A_8}^1 = 0.$$

By Theorem 3, Theorem 5, Theorem 7 and Theorem 8,

$$(1_{A_9}, 36 \downarrow_{A_9}^{A_{10}})_{A_9}^1 = (1_{A_9}, 36 \downarrow_{A_9}^{A_{10}} \oplus 27_{A_9})_{A_9}^1 = (1_{A_9}, 7 \uparrow_{A_8}^{A_9})_{A_9}^1 = (1_{A_8}, 7_{A_8})_{A_8}^1 = 0.$$

$$(1_{A_9}, 90 \downarrow_{A_9}^{A_{10}})_{A_9}^1 = (1_{A_9}, 90 \downarrow_{A_9}^{A_{10}} \oplus 162_{A_9})_{A_9}^1 = (1_{A_9}, 28 \uparrow_{A_8}^{A_9})_{A_9}^1 = (1_{A_8}, 28_{A_8})_{A_8}^1 = 0.$$

$$(1_{A_9}, 126 \downarrow_{A_9}^{A_{10}})_{A_9}^1 = (1_{A_9}, 126 \downarrow_{A_9}^{A_{10}} \oplus 189_{A_9})_{A_9}^1 = (1_{A_9}, 35 \uparrow_{A_8}^{A_9})_{A_9}^1 = (1_{A_8}, 35_{A_8})_{A_8}^1 = 1.$$

$$(1_{A_9}, 279 \downarrow_{A_9}^{A_{10}})_{A_9}^1 = (1_{A_9}, 13 \uparrow_{A_8}^{A_9})_{A_9}^1 = (1_{A_8}, 13_{A_8})_{A_8}^1 = 1. \quad \square$$

A similar proof gives

Lemma 2.

$$(36, S)_{A_{10}}^1 = \begin{cases} 1, & \text{if } S \in \{126, 279\}; \\ 0, & \text{otherwise.} \end{cases}$$

$$(90, S)_{A_{10}}^1 = \begin{cases} 1, & \text{if } S = 126 \\ 0, & \text{otherwise.} \end{cases}$$

$$(126, S)_{A_{10}}^1 = \begin{cases} 1, & \text{if } S \in \{9, 36, 90\}; \\ 0, & \text{otherwise.} \end{cases}$$

$$(279, S)_{A_{10}}^1 = \begin{cases} 1, & \text{if } S \in \{9, 36\}; \\ 0, & \text{otherwise.} \end{cases}$$

4.2. Principal block.

Lemma 3.

$$(1, S)_{A_{10}}^1 = \begin{cases} 1, & \text{if } S \in \{34, 41, 84, 224\}; \\ 0, & \text{otherwise.} \end{cases}$$

$$(34, S)_{A_{10}}^1 = \begin{cases} 1, & \text{if } S \in \{1, 41, 224\}; \\ 0, & \text{otherwise.} \end{cases}$$

$$(41, S)_{A_{10}}^1 = \begin{cases} 1, & \text{if } S \in \{1, 34, 84, 224\}; \\ 0, & \text{otherwise.} \end{cases}$$

$$(84, S)_{A_{10}}^1 = \begin{cases} 1, & \text{if } S \in \{1, 41, 84\}; \\ 0, & \text{otherwise.} \end{cases}$$

$$(224, S)_{A_{10}}^1 = \begin{cases} 1, & \text{if } S \in \{1, 34, 41, 224\}; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The difficulty arises only when computing $(84, S)_{A_{10}}^1$ (noting that, since all the simple FA_{10} -modules are self-dual, $(S, T)_{A_{10}}^1 = (T, S)_{A_{10}}^1$). In all other cases, a similar proof to that in the case of non-principal blocks will suffice.

By block theory, Theorem 3, Theorem 5, Theorem 7 and Theorem 8,

$$(84, 1)_{A_{10}}^1 = (84, 1 \uparrow_{A_9}^{A_{10}})_{A_{10}}^1 = (84 \downarrow_{A_9}, 1_{A_9})_{A_9}^1 = (\mathfrak{D}(21; 7, 35; 21), 1)_{A_9}^1.$$

Furthermore, Theorem 6 implies that there are only three possibilities:

$$\begin{array}{c} 21 \\ \swarrow \quad \searrow \\ 7 \quad 35 \\ \swarrow \quad \searrow \\ 1 \quad 21 \end{array}, \quad \begin{array}{c} 21 \\ \swarrow \quad \searrow \\ 7 \quad 35 \\ \swarrow \quad \searrow \\ 1 \quad 21 \end{array}, \quad \begin{array}{c} 21 \\ \swarrow \quad \searrow \\ 7 \quad 35 \\ \swarrow \quad \searrow \\ 21 \quad 1 \end{array}$$

However, there is a unique copy of 1 in $L_3(P_{21A_9})$ and the first structure exists by the submodule lattice of $21 \uparrow_{A_8}^{A_9} \cdot f_0$ obtained by the MeatAxe. Therefore, we see that the other two cases cannot occur (see 6.2.2).

The same proof gives $(84, 41)_{A_{10}}^1 = 1$.

Now using appropriate long exact sequences of cohomology, we have

$$(84, 84)_{A_{10}}^1 \geq 1; \quad (84, 34)_{A_{10}}^1 \leq 1; \quad (84, 224)_{A_{10}}^1 \leq 1;$$

We will finish determining the dimensions of the above Ext-spaces in Section 6.2.1 and Section 6.2.2. \square

5. INDUCTION OF 2-STEP FA_9 - MODULES

In this section we shall induce to A_{10} each of the non-trivial extensions of a simple module by a simple module for A_9 .

By Frobenius reciprocity, we have

$$\begin{pmatrix} 1 \\ 7 \end{pmatrix} \uparrow_{A_9}^{A_{10}} .e_0 = \frac{1}{34} \quad , \quad \begin{pmatrix} 1 \\ 41 \end{pmatrix} \uparrow_{A_9}^{A_{10}} .e_0 = \frac{1}{41} \quad , \quad \begin{pmatrix} 1 \\ 35 \end{pmatrix} \uparrow_{A_9}^{A_{10}} .e_0 = \frac{1}{224} \quad ,$$

$$\begin{pmatrix} 7 \\ 35 \end{pmatrix} \uparrow_{A_9}^{A_{10}} .e_0 = \frac{34}{224} \quad , \quad \begin{pmatrix} 7 \\ 41 \end{pmatrix} \uparrow_{A_9}^{A_{10}} .e_0 = \frac{34}{41} \quad ,$$

$$\begin{pmatrix} 35 \\ 35 \end{pmatrix} \uparrow_{A_9}^{A_{10}} .e_0 = \frac{224}{224} \quad , \quad \begin{pmatrix} 35 \\ 41 \end{pmatrix} \uparrow_{A_9}^{A_{10}} .e_0 = \frac{224}{41} \quad ,$$

$$\begin{pmatrix} 21 \\ 7 \end{pmatrix} \uparrow_{A_9}^{A_{10}} .e_0 = \frac{84}{84} \frac{41}{34} \quad , \quad \begin{pmatrix} 21 \\ 35 \end{pmatrix} \uparrow_{A_9}^{A_{10}} .e_0 = \frac{84}{84} \frac{41}{224} \quad .$$

We shall determine the structure of $\begin{pmatrix} 27 \\ 189 \end{pmatrix} \uparrow_{A_9}^{A_{10}} .e_0$ and $\begin{pmatrix} 189 \\ 27 \end{pmatrix} \uparrow_{A_9}^{A_{10}} .e_0$ in Section 6.2.

 6. STRUCTURE OF PROJECTIVE MODULES FOR FA_{10}

6.1. The non-principal block. We will use Section 4.1 and self-duality to find the structure of P_{90}, P_{279}, P_9 and P_{36} . For P_{126} , we will additionally need Landrock's lemma.

The Cartan matrix shows that $P_{90A_{10}}$ has composition factors $9 + 36 + 3(90) + 2(126)$.

Also, it has both head and socle isomorphic to 90 i.e. $P_{90A_{10}} = \begin{matrix} 90 \\ X \\ 90 \end{matrix}$ where X has composition factors $9 + 36 + 90 + 2(126)$. Section 4.1 and self-duality shows that $L_1(X) =$

$S_1(X) = 126$. Hence, $X = \begin{matrix} 126 \\ 9 \quad 36 \quad 90 \end{matrix}$, and $P_{90A_{10}}$ has Loewy structure as claimed in Theorem 2.

• Similarly, $P_{279A_{10}} = \begin{matrix} 279 \\ Y \\ 279 \end{matrix}$ where Y has composition factors $2(9) + 2(36) + 126 + 279$, and $L_1(Y) = S_1(Y) = 9 \oplus 36$. Therefore, the structure of Y can only be one of the following:

$$\begin{matrix} S \\ 126 \quad 279 \quad T \oplus T \\ S \end{matrix} \quad \text{or} \quad \begin{matrix} 9 \quad 36 \\ 126 \quad 279 \\ 9 \quad 36 \end{matrix} .$$

Here $\{S, T\} = \{9, 36\}$. But the first case cannot happen since $(S, T)_{A_{10}}^1 = 0$ by Section 4.1. Thus, the structure of $P_{279A_{10}}$ is as claimed in Theorem 2.

• We will work out in details the structure of $P_{9A_{10}}$. The structure of $P_{36A_{10}}$ can be found in a similar way.

The Cartan matrix shows that $P_{9A_{10}} = \begin{matrix} 9 \\ Z \\ 9 \end{matrix}$ where Z has composition factors $2(9) + 36 + 90 + 2(126) + 2(279)$. It is easy to see that $L_1(Z) = S_1(Z) = 126 \oplus 279$. Consequently, the structure of Z can only be one of the following:

$$\begin{array}{ccccccc}
& & & & Q & & 126 \quad 279 \\
& & Q & & 9 & & 126 \quad 279 \\
9 & 9 & 36 & 90 & R \oplus R & \text{or} & 36 & 90 & R \oplus R & \text{or} & 9 & 9 & 36 & 90 & \text{or} & 36 & 90 & . \\
& & Q & & 9 & & 126 & 279 & & & 9 & & 126 & 279 & & & & \\
& & & & Q & & 126 & 279
\end{array}$$

Here $\{R, Q\} = \{126, 279\}$. But the first case cannot happen since $(R, Q)_{A_{10}}^1 = 0$, while the second and the fourth are impossible since $(9, 36)_{A_{10}}^1 = 0$ by Section 4.1. Hence, the structure of $P_{9A_{10}}$ is as claimed in Theorem 2.

• By Landrock's lemma and the structure of the PIMs obtained so far, $P_{126A_{10}}$ has one copy of 9, one copy of 36 and one copy of 90 in $L_2(P_{126A_{10}})$ and $L_4(P_{126A_{10}})$ each. It also has one copy of 279 in $L_3(P_{126A_{10}})$. Now self-duality forces $P_{126A_{10}}$ to have the claimed structure in Theorem 2.

6.2. The principal block.

6.2.1. *The Loewy structure of P_{34} and P_{224} .* Using Theorem 8, we have the following filtrations:

$$P_{34A_{10}} = P_{27A_9} \uparrow^{A_{10}} .e_0 = \begin{pmatrix} 27 \\ 189 \\ 27 \end{pmatrix} \uparrow .e_0$$

$$\sim \begin{array}{c|c|c}
\begin{array}{c} 34 \\ 1 \quad 41 \\ 34 \quad 84 \\ 1 \quad 41 \\ 34 \end{array} & \begin{array}{c} 224^* \\ 1_a \quad 41_a \quad 224_a \\ 1_b \quad 224_b \quad 34_b \quad 84_b \\ 1_c \quad 41_c \quad 224_c \\ 224 \end{array} & \begin{array}{c} 34_a \\ 1_f \quad 41_f \\ 34_f \quad 84_f \\ 1 \quad 41 \\ 34 \end{array}
\end{array}$$

Figure 1.

$$P_{224A_{10}} = P_{189A_9} \uparrow^{A_{10}} .e_0 = \begin{pmatrix} 189 \\ 27 \\ 189 \end{pmatrix} \uparrow .e_0$$

$$\sim \begin{array}{c|c|c}
\begin{array}{c} 224 \\ 1 \quad 41 \quad 224 \\ 1 \quad 224 \quad 34 \quad 84 \\ 1 \quad 41 \quad 224 \\ 224_f \end{array} & \begin{array}{c} 34 \\ 1_e \quad 41_e \\ 34_d \quad 84_d \\ 1 \quad 41 \\ 34_e \end{array} & \begin{array}{c} 224_d \\ 1 \quad 41 \quad 224_g \\ 1_h \quad 224_h \quad 34_g \quad 84_h \\ 1_g \quad 41_g \quad 224_e \\ 224 \end{array}
\end{array}$$

Figure 2.

Figures 1 and 2 give $(34, 84)_{A_{10}}^1 = (224, 84)_{A_{10}}^1 = 0$ as claimed in Lemma 3.

Section 4.2 implies $224^* \in L_2(P_{34A_{10}})$. Figure 2 and Section 4.2 show that there is exactly one copy of 34 in $L_3(P_{224A_{10}})$. Hence, in Figure 1, $224_a \in L_3(P_{34A_{10}})$ by Landrock's lemma. Moreover, 34_b extends 224_a by the structure of $189 \uparrow$, and hence it can only be in $L_4(P_{34A_{10}})$, $L_5(P_{34A_{10}})$ or $L_6(P_{34A_{10}})$. $(34, S)_{A_{10}}^1$ implies it can only be in $L_4(P_{34A_{10}})$.

This, in turn, forces $1_a, 41_a \in L_3(P_{34A_{10}})$ and

$$(*) \quad 1_c, 41_c, 224_c \notin L_6(P_{34A_{10}})$$

by the structure of $189 \uparrow$ and Section 4.2 respectively.

Now 84_b is not in either $L_5(P_{34_{A_{10}}})$ or $L_6(P_{34_{A_{10}}})$ by Section 4.2 and the Benson-Carlson diagram of $189 \uparrow$, so it has to be in $L_4(P_{34_{A_{10}}})$. Moreover, Section 4.2 and $189 \uparrow$ also imply $224_b \notin L_5(P_{34_{A_{10}}})$. Also, $224_b \notin L_6(P_{34_{A_{10}}})$: otherwise, there exists a submodule U of $\left(\begin{smallmatrix} 27 \\ 189 \end{smallmatrix}\right) \uparrow_{A_9}^{A_{10}}$ of Loewy length 4 with simple head 34 and simple socle 224. Hence, by duality, there exists a subquotient U^* of $\left(\begin{smallmatrix} 189 \\ 27 \end{smallmatrix}\right) \uparrow_{A_9}^{A_{10}}$ with simple head 224, simple socle 34 and Loewy length 4. This means that, in Figure 2, $34_d \in L_4(P_{224_{A_{10}}})$, since Section 4.2 forces $224_d \in L_i(P_{224_{A_{10}}})$ for $i \geq 3$ so the diagram of $189 \uparrow$ implies $34_g \notin L_4(P_{224_{A_{10}}})$. This in turn forces $1_e, 41_e \in L_3(P_{224_{A_{10}}})$. Hence, $84_d \in L_4(P_{224_{A_{10}}})$ since $(224, 84)_{A_{10}}^1 = 0$. Therefore, we have

$$\text{Loewy length of } \left(\begin{smallmatrix} 189 \\ 27 \end{smallmatrix}\right) \uparrow_{A_9}^{A_{10}} \cdot e_0 \leq 7 < 8 \leq \text{Loewy length of } \left(\begin{smallmatrix} 27 \\ 189 \end{smallmatrix}\right) \uparrow_{A_9}^{A_{10}} \cdot e_0,$$

a contradiction. Therefore, $224_b \in L_4(P_{34_{A_{10}}})$.

Next, Landrock's Lemma implies that $34_d \in L_4(P_{224_{A_{10}}})$. The same argument as above shows that Loewy length of $\left(\begin{smallmatrix} 189 \\ 27 \end{smallmatrix}\right) \uparrow_{A_9}^{A_{10}} \cdot e_0$ is at most 7, hence $1_b \notin L_6(P_{34_{A_{10}}})$. Moreover, $1_b \notin L_5(P_{34_{A_{10}}})$ by the Benson-Carlson diagram of $189 \uparrow \cdot e_0$ and Section 4.2. Therefore, $1_b \in L_4(P_{34_{A_{10}}})$. Now (*) gives

$$\left(\begin{smallmatrix} 27 \\ 189 \end{smallmatrix}\right) \uparrow_{A_9}^{A_{10}} \cdot e_0 = \begin{array}{ccccccc} & & & & 34 & & \\ & & & & 1 & 41 & 224 \\ & & & & 34 & 84 & 1 & 41 & 224 \\ & & & & 1 & 41 & 1 & 224 & 34 & 84 \\ & & & & 34 & 1 & 41 & 224 & & \\ & & & & 224 & & & & & \end{array}.$$

This has Loewy length 6, so Loewy length of $\left(\begin{smallmatrix} 189 \\ 27 \end{smallmatrix}\right) \uparrow_{A_9}^{A_{10}} \cdot e_0 = 6$. This implies $34_e \in L_6(P_{224_{A_{10}}})$. The diagram of $27 \uparrow$ now gives

$$\left(\begin{smallmatrix} 189 \\ 27 \end{smallmatrix}\right) \uparrow_{A_9}^{A_{10}} \cdot e_0 = \begin{array}{ccccccc} & & & & 224 & & \\ & & & & 1 & 41 & 224 & 34 \\ & & & & 1 & 224 & 34 & 84 & 1 & 41 \\ & & & & 1 & 41 & 224 & 34 & 84 & \\ & & & & 224 & 1 & 41 & & & \\ & & & & 34 & & & & & \end{array}.$$

Now, in Figure 1, $34_a \notin L_2(P_{34_{A_{10}}})$ by Section 4.2. In fact, it is in $L_3(P_{34_{A_{10}}})$, otherwise, it must extend (and indeed lie below) some composition factors in the submodule $\begin{array}{cc} 1 & 41 \\ & 34 \end{array}$ of $27 \uparrow$, which is impossible.

Also, Section 4.2 implies that $34_f, 84_f \notin L_6(P_{34_{A_{10}}})$; and they are not in $L_7(P_{34_{A_{10}}})$ either, because the diagram of $27 \uparrow$ and Section 4.2 imply $1_f, 41_f \notin L_6(P_{34_{A_{10}}})$. Thus, $34_f, 84_f \in L_5(P_{34_{A_{10}}})$. This, in turns, forces $1_f, 41_f \in L_4(P_{34_{A_{10}}})$. Therefore, we obtain the structure of $P_{34_{A_{10}}}$ as claimed in Theorem 1.

Finally we compute the structure of $P_{224_{A_{10}}}$. By Landrock's lemma, there is one copy of 34 in $L_5(P_{224_{A_{10}}})$, so $34_g \in L_5(P_{224_{A_{10}}})$. Thus, $224_d \in L_3(P_{224_{A_{10}}})$ by the diagram of $189 \uparrow$. Also, since 84 does not extend 224, $84_h \in L_4(P_{224_{A_{10}}})$.

6.2.2. *The Loewy structure of P_{84} .* We have

Similarly,

Furthermore, we have the following filtration ((3) in (Siegel, 1991)):

Finally, the structure of $P_{34A_{10}}$ together with the above data gives the following Loewy structure:

$$\begin{array}{cccccccccccccccc}
& & & & & & 34 & 84 & & & & & & & & & \\
& & & & & 1 & 41 & 1 & 41 & 224 & 84 & & & & & & \\
& 34 & 84 & 224 & 224 & 1 & 41 & 34 & 84 & 34 & 84 & 1 & 41 & & & & \\
= & 41 & 224 & 84 & 1 & 41 & 1 & 1 & 41 & 1 & 34 & 1 & 34 & 41 & 224 & 84 & . \\
& & 34 & 84 & 34 & 84 & 1 & 1 & 224 & 224 & 1 & 41 & 34 & 84 & & & \\
& & & & & 41 & 224 & 84 & 1 & 41 & 1 & & & & & & \\
& & & & & & & & & 34 & 84 & & & & & &
\end{array}$$

Thus, the structure of $P_{84A_{10}}$ is as claimed in Theorem 1 and $(84, 84)_{A_{10}}^1 = 1$ as claimed in Section 4.

6.2.3. *The Loewy structure of P_{41} .* Landrock's Lemma, Section 6.2.1, Section 6.2.2 and the filtration of $P_{41A_9} \uparrow^{A_{10}} .e_0$ give the positions of the unlabelled simple modules and 1_a in the Loewy structure for $P_{41A_{10}}$ as follows:

$$\begin{array}{cccccccc}
 & & & & 41 & & & \\
 & & & & 1_a & 34 & 84 & 224 \\
 & & 34 & 224 & 84 & 1_A & 41_A & 41_D & 41_E & 41_F \\
 34 & 34 & 224 & 224 & 84 & 84 & 1_B & 41_B & 1_D & 1_H \\
 & 34 & 224 & 84 & 1_C & 41_C & 41_G & 1_G & 41_H \\
 & & & 34 & 224 & 84 & 1_E \\
 & & & & 1_F & 41_E \\
 & & & & & 41_F
 \end{array}$$

Since we have accounted for all the 84s by Landrock's Lemma and the 84s all come from $\mathfrak{D}(84; 1; 41; 84)$, $1_A, 41_A, 1_B, 41_B, 1_C, 41_C$ are in the respective Loewy layers as above.

Next, from (Siegel, 1991), we see that

$$\begin{pmatrix} 13 \\ 1 \end{pmatrix} \uparrow_{A_8}^{A_9} .f_0 = \begin{array}{cc} 41 & \\ 35 & 1 \\ 41 & 7 \\ 1 & \end{array}$$

Thus, $M := \left(\begin{pmatrix} 13 \\ 1 \end{pmatrix} \uparrow_{A_8}^{A_9} .f_0 \right) \uparrow^{A_{10}} = \begin{array}{cc} 41 & \\ 224 & 1 \\ 41_a & 34_a \\ 1_B & \end{array}$, since Frobenius Reciprocity and Lemma 3

implies $S_1(M) = 1$ and $41_a, 34_a \notin L_1(M), L_2(M)$. Therefore, we get a copy of 41 and a copy of 1 in $L_3(P_{41A_{10}})$ that do not come from $21 \uparrow_{A_9}^{A_{10}} = \mathfrak{D}(84; 1; 41; 84)$, say 41_D and 1_D respectively.

Next, Frobenius Reciprocity gives

$$\begin{pmatrix} 41 \\ 1 & 7 \\ 41 \end{pmatrix} \uparrow_{A_9}^{A_{10}} .e_0 = \begin{array}{cc} 41 & \\ 1 & 34 \\ 41 & \end{array}$$

so 1_a extends a 41_E in $L_3(P_{41A_{10}})$, and $41_E \neq 41_A, 41_D$ since $13 \uparrow_{A_8}^{A_9} = \begin{array}{c} 41 \\ 35 \\ 41_1 \end{array}$ and there

exists an FA_9 -module $\begin{array}{c} 41 \\ 1 & 7 \\ 41_2 \end{array}$. Hence, $41_1 \neq 41_2$ because there is only one copy of 35

in $L_2(P_{41A_9})$. Thus, 41_E comes from $41_2 \uparrow_{A_9}^{A_{10}}$, 41_D comes from $41_1 \uparrow_{A_9}^{A_{10}}$ and 41_A comes from $\mathfrak{D}(84; 1; 41; 84)$.

Note that all the factors in $L_2(P_{41A_{10}})$ extend some factors in $L_3(P_{41A_{10}})$ so self-duality implies that the layer immediately above the unique bottom 41 in $P_{41A_{10}}$ is

$$\begin{array}{cccc}
 1 & 34 & 84 & 224
 \end{array}$$

Therefore, the Loewy length of $P_{41A_{10}}$ is 7 and there is exactly one copy of 1 in $L_6(P_{41A_{10}})$, say 1_E . We can also fill in the bottom 41, say 41_F .

Now 41_D extends 1_D so by self-duality and the submodule lattice of $P_{41A_{10}}$, we must have another copy of 41 in $L_5(P_{41A_{10}})$, say 41_G .

$$(*) \quad M.rad(M)^r \subseteq soc(M)^{n-r}$$

Now inducing the first three Loewy layers of $P_{41_{A_9}}$ to A_{10} , we get the following filtration:

[illegible]

Next, by Section 2, Theorem 8 and Frobenius Reciprocity, we have

$$X = \begin{pmatrix} 28 \\ 35 \\ 1 \end{pmatrix} \uparrow_{A_8}^{A_9} . f_0 = \begin{pmatrix} 41 \\ 35 & 1 & 7 \\ 21 & 35 & 41 & 1 \\ 35 & 7 \\ 1 \end{pmatrix} \text{ and } X \uparrow^{A_{10}} . e_0 = \begin{matrix} & 41 \\ & 224 & 1 & 34 & 84 \\ 224 & 41 & 1 & 1 & 41 \\ & 34_a & 224_a & 84 \\ & & 1 \end{matrix},$$

Similarly, $Y = \begin{pmatrix} 28 \\ 35 \end{pmatrix} \uparrow_{A_8}^{A_9} . f_0 = \begin{pmatrix} 41 \\ 35 & 1 & 7 \\ 21 & 35 & 41 \\ 35 & \diagdown & \end{pmatrix}$ and $Y \uparrow_{A_{10}}^{A_9} . e_0 = \begin{matrix} & & 41 \\ & 224 & 1 & 34 & 84 \\ 41 & \diagdown & 224 & 1 & 41 \\ & 224 & 84 \end{matrix}$.

The remaining copy of 1 must be in $L_3(P_{41_{A_{10}}})$ or $L_4(P_{41_{A_{10}}})$ by (*) and Section 4. Using the filtration (6) in (Siegel, 1991) below and the fact that $P_{41_{A_{10}}} = P_{41_{A_9}} \uparrow_{A_9}^{A_{10}} .e_0$, we see that the remaining 1 must be in $L_4(P_{41_{A_{10}}})$ and the structure of $P_{41_{A_{10}}}$ is as claimed in Theorem 1.

$$P_{41_{A_9}} = P_{13_{A_8}} \uparrow^{A_9} . f_0$$

$$\sim \left| \begin{array}{ccc} 41 & & \\ 35 & & \\ 41 & & \end{array} \right| \left| \begin{array}{ccc} 1 & & 7 \\ 7 & \oplus 1 & 21 \\ 1 & & 7 \end{array} \right| \left| \begin{array}{ccc} 41 & & \\ 35 & & \\ 41 & & \end{array} \right| \left| \begin{array}{ccc} 35 & & \\ 21 & \oplus 35 & \\ 35 & & \end{array} \right| \left| \begin{array}{ccc} 1 & & 7 \\ 7 & \oplus 1 & 21 \\ 1 & & 7 \end{array} \right| \left| \begin{array}{ccc} 41 & & \\ 35 & & \\ 41 & & \end{array} \right|$$

positions of the four 1^* . By (*) in Section 6.2.3, at least one of the 1^* must appear in $L_4(P_{1_{A_{10}}})$. Using the filtration (2) in Siegel (1991) below, and that $P_{1_{A_{10}}} = P_{1_{A_9}} \uparrow^{A_{10}} .e_0$, the remaining three 1^* must be in $L_5(P_{1_{A_{10}}})$ and the structure of $P_{1_{A_{10}}}$ is as claimed in Theorem 1.

$$P_{1_{A_9}} = P_{1_{A_8}} \uparrow^{A_9} .f_0$$

$$\sim \begin{array}{c} 1 \\ 7 \\ 1 \end{array} \left| \begin{array}{cc} 41 & 35 \\ 35 \oplus 21 & 35 \\ 41 & 35 \end{array} \right| \begin{array}{cccc} 1 & 1 & 7 & 41 \\ 7 \oplus 7 \oplus 1 & 21 \oplus 1 & 7 & 41 \\ 1 & 1 & 7 & 41 \end{array} \left| \begin{array}{cc} 41 & 35 \\ 35 \oplus 21 & 35 \\ 41 & 35 \end{array} \right| \begin{array}{c} 1 \\ 7 \\ 1 \end{array}$$

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APPENDIX

Decomposition matrix of A_{10} mod 3

[illegible]

Cartan matrix

[illegible]

$\dim_F \text{Ext}_{A_{10}}^1(S, T)$ for S, T simple

	1	34	41	84	224	9	36	90	126	279	567
1	.	1	1	1	1
34	1	.	1	.	1
41	1	1	.	1	1
84	1	.	1	1
224	1	1	1	.	1
9	1	1	.
36	1	1	.
90	1	.	.
126	1	1	1	.	.	.
279	1	1
567

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